

Maximizing Submodular Set Functions

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Submodularity and submodular set functions play a fundamental and unifying role in combinatorial optimization. A cornerstone result is a strongly polynomial-time algorithm for submodular function *minimization*. It was also realized early on that submodular set function *maximization* has several applications, notably for monotone functions with matroid constraints. However, most of the maximization problems of interest are NP-hard. The work of Cornuejols, Fisher, Nemhauser, and Wolsey in the late 70's analyzed a simple greedy algorithm and also local search in the framework of *approximation* algorithms.

In the last twenty years there has been a resurgence of interest in understanding various aspects of submodular set function maximization (and also minimization) including approximation algorithms, lower bounds, and applications. The impetus for this renewed interest came from both theoretical developments and applications in several areas including algorithmic game theory, data analysis, machine learning, stochastic optimization, and others.

The purpose of this manuscript is to bring together, in one place, a number of new, exciting, and useful results that have been obtained since 2005. The classical greedy algorithm has been extended and refined in various ways. Local-search has been shown to yield a number of new results, especially for nonnegative functions. A mathematical programming methodology based on the multilinear relaxation has provided a powerful new methodology along with novel tools. The symmetry gap technique has provided a unified way to prove lower bounds. The combination of the past and recent techniques have yielded a number of new algorithms and hardness results, tight results in some fundamental settings, and new structural results such as correlation gaps and concentration bounds. This manuscript focuses on polynomial-time approximation algorithms for nonnegative submodular functions and associated technical ideas.

Contents

1	Introduction and Preliminaries	1
1.1	Basics	1
1.2	Matroids	3
1.3	Examples of Submodular Functions	4
1.4	Constrained Submodular Function Maximization	6
1.4.1	Independence family constraints	7
1.5	A Brief Overview of Algorithmic Techniques	10
1.5.1	A Few Highlights	12
1.6	Guide to the rest of the book	13
2	Greedy Algorithm and Variants	17
2.1	Monotone Submodular Functions	19
2.1.1	Cardinality Constraint	19
2.1.2	Knapsack Constraint	22
2.1.3	Matroid Constraint	26
2.1.4	Local-Greedy for Partition Matroid Constraint	28
2.1.5	Multiple Matroids and p -System Constraints	30
2.2	Nonnegative Submodular Functions	33
2.2.1	Unconstrained nonnegative maximization	33
2.2.2	Iterated Greedy for Nonnegative Functions	35
2.2.3	Simultaneous Greedys	38
2.3	Randomized Greedy for Nonnegative Functions	39
2.4	Greedy with preemption	42
2.5	Notes and Bibliographic Remarks	46
3	Local Search	49
3.1	Unconstrained nonnegative submodular maximization	52
3.2	Matroid and p -exchange constraints	55
3.2.1	Single Matroid Constraint	55

3.2.2	Matroid base constraint	58
3.2.3	Multiple matroid constraints	60
3.3	Improved approximations and a general perspective	63
3.3.1	A polyhedral perspective	65
3.3.2	Improvement for Matroid Intersection	68
3.4	Notes and bibliographic remarks	70
4	The Double Greedy Algorithm	73
4.1	Deterministic double-greedy algorithm	74
4.2	Randomized double-greedy algorithm	75
4.3	Notes and Bibliographic Remarks	78
5	Multilinear Relaxation	79
5.1	Introducing the Multilinear Relaxation	80
5.1.1	Definitions	81
5.1.2	Convexity Properties	82
5.1.3	Evaluation	86
5.2	The Continuous Greedy Algorithm	86
5.3	Continuous Greedy for Nonmonotone Functions	91
5.3.1	Measured Continuous Greedy	94
5.4	Local Search	96
5.5	Adapting Continuous Greedy for Additional Applications	97
5.5.1	Maximizing multiple submodular functions	98
5.5.2	Comparing with other extensions of f	99
5.6	Notes and Bibliographic Remarks	101
6	Rounding in the Matroid Polytope	105
6.1	Randomized pipage rounding	108
6.2	Randomized swap rounding	112
6.3	Pipage rounding for the Knapsack polytope	115
6.4	Negative correlation and concentration properties	116
6.4.1	Concentration properties under independent rounding	116
6.4.2	Concentration under matroid rounding	118
6.5	Notes and Bibliographic Remarks	122
7	Nonoblivious Local Search for Matroids	125
7.1	A Randomized Algorithm	125
7.1.1	Auxiliary function	126
7.1.2	Algorithm and analysis	127
7.2	A Deterministic Algorithm	130

7.3	Notes and Bibliographic Remarks	135
8	Rounding via Contention Resolution Schemes	137
8.1	Definition of Contention Resolution Schemes	139
8.1.1	Submodular maximization via CR schemes	140
8.2	Implementations of Contention Resolution Schemes	145
8.2.1	Contention Resolution for Matroid Polytopes	152
8.2.2	An optimal CR scheme for a matroid via a general framework	154
8.2.3	Contention Resolution for Packing Integer Programs	158
8.3	Illustrating applications of the framework	164
8.4	Online Contention Resolution Schemes	167
8.5	Notes and Bibliographic Remarks	169
9	Allocation, orienteering, and routing problems	171
9.1	Allocation problems	171
9.1.1	Welfare maximization	172
9.1.2	Generalized and Separable Assignment Problems	174
9.1.3	Max-min Welfare	178
9.1.4	Nash social welfare	183
9.2	Submodular Orienteering and Chains in a Poset	192
9.3	Routing problems via LP-based algorithms	196
9.4	Notes and Bibliographic Remarks	199
10	Hardness results for submodular maximization	203
10.1	NP-hardness from reductions and direct oracle hardness	203
10.2	Oracle hardness from the symmetry gap	206
10.2.1	Optimality of 1/2-approximation for unconstrained submodular maximization	207
10.2.2	Other symmetry gap examples	210
10.2.3	Hardness of solving the multilinear relaxation	211
10.2.4	k -extendible systems	213
10.3	Modular vs Submodular Function Maximization	213
10.4	Notes and Bibliographic Remarks	217
11	A Quick Survey of Other Topics	219
11.1	Faster algorithms	219
11.1.1	Speeding up combinatorial algorithms	220
11.1.2	Speeding up relaxation-based algorithms	221
11.2	Beyond the sequential model of computation	222

11.2.1	Parallel, Distributed, and Low-adaptivity Algorithms .	223
11.2.2	Streaming Algorithms	225
11.2.3	Online Algorithms	227
11.2.4	Dynamic Algorithms	228
11.3	Robustness	229
11.4	Restrictions and generalizations of submodular functions . . .	229
12	A Selection of Open Problems	231
A	Continuous Extensions of Submodular Functions	237
A.1	Relating the extensions	241
A.2	Notes and Bibliographic Remarks	248
B	Subclasses and Generalizations of Submodular Functions	251
B.1	Subclasses of submodular functions	251
B.2	Complement-free classes generalizing submodular functions .	255
B.3	Beyond complement-free classes	256
B.4	Beyond binary domains	258
B.5	Classes of continuous functions	259