

Exercise Set 2

Exercise 2.1. Let $G = (A \dot{\cup} B, E)$ be a bipartite graph with $n = |A| \geq |B|$, and consider the following algorithm for maximum matching. For an edge set $F \subseteq E$, and let $d_F(v)$ denote the number of edges incident to $v \in A \dot{\cup} B$.

Throughout, we maintain $F \subseteq E$ such that $d_F(v) = 1$ for all $v \in B$ (but vertices in A may have arbitrary degree). Further, we maintain labels $h : A \rightarrow \{0, 1, 2, \dots, n\}$. Initially, we select F by picking for each $v \in B$ an arbitrary edge incident to v , and set $h(u) = 0$ for every $u \in A$.

In every step of the algorithm, if $d_F(u) \leq 1$ for all $u \in A$, then we terminate by returning F . Otherwise, we select a vertex $u \in A$ with $d_F(u) \geq 2$ and $h(u) < n$. If there exists $(u, v) \in F$ and $(w, v) \in E \setminus F$ such that $h(u) = h(w) + 1$, then we replace F by $(F \setminus \{(u, v)\}) \cup \{(w, v)\}$. If no such (u, v) and (w, v) exist, then we increase $h(u)$ by 1.

Finally, if $h(u) = n$ for all vertices $u \in A$ with $d_F(u) \geq 2$, then we remove from F arbitrarily all but one edges incident to the nodes $\{u : h(u) = n\}$, and return the remaining edges from F .

- (a) Show that in every step of the algorithm, $d_F(v) = 1$ for all $v \in B$. If $d_F(u) = 0$ for $u \in A$, then $h(u) = 0$.
- (b) Show that in every step of the algorithm, for every $u, w \in A$, $v \in B$ such that $(u, v) \in F$ and $(w, v) \in E \setminus F$, it holds that $h(u) \leq h(w) + 1$.
- (c) Assume at some point of the algorithm there is a node $u \in A$ with $d_F(u) > 1$ and $h(u) = n$. Show that there exists a value $r \in \{0, 1, \dots, n\}$ such that there exists no $u' \in A$ with $h(u') = r$. Further, let $Y = \{u \in A \mid h(u) > r\}$, and let $X = \{v \in B \mid \exists u \in Y, (u, v) \in F\}$ denote the set of items in B that are mapped by F to Y . Show that $\Gamma(X) = Y$, and therefore X violates Hall's condition.
- (d) Show that the set F returned by the algorithm is a maximum matching.
- (e) What is the best running time bound you can come up with for this algorithm?

(1+3+3+3+2 points)

Exercise 2.2. Let G be a 3-regular undirected graph.

- (a) Assume G is simple. Show that there is a matching in G covering at least $(7/8) \cdot |V(G)|$ vertices.
- (b) Give an example to prove that the bound of item (a) is tight.
- (c) Show that the assumption that G is simple in item (a) is necessary.

(2+1+1 points)

Deadline: October 27th, before the lecture. The websites for lecture and exercises can be found at:

https://ecampus.uni-bonn.de/ilias.php?baseClass=ilrepositorygui&cmdNode=z6:n8&cmdClass=ilObjCourseGUI&ref_id=4281645

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